

Effective interactions, Fermi-Bose duality, and ground states of ultracold atomic vapors in tight de Broglie waveguides

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Derivation of effective zero-range one-dimensional (1D) interactions between atoms in tight waveguides is reviewed, as is the Fermi-Bose mapping method for determination of exact and strongly-correlated many-body ground states of ultracold bosonic and fermionic atomic vapors in such waveguides, including spin degrees of freedom. Odd-wave 1D interactions derived from 3D p-wave scattering are included as well as the usual even-wave interactions derived from 3D s-wave scattering, with emphasis on the role of 3D Feshbach resonances for selectively enhancing s-wave or p-wave scattering so as to reach 1D confinement-induced resonances of the even and odd-wave interactions. A duality between 1D fermions and bosons with zero-range interactions suggested by Cheon and Shigehara is shown to hold for the effective 1D dynamics of a spinor Fermi gas with both even and odd-wave interactions and that of a spinor Bose gas with even and odd-wave interactions, with even(odd)-wave Bose coupling constants inversely related to odd(even)-wave Fermi coupling constants. Some recent applications of Fermi-Bose mapping to determination of many-body ground states of Bose gases and of both magnetically trapped, spin-aligned and optically trapped, spin-free Fermi gases are described, and a new generalized Fermi-Bose mapping is used to determine the phase diagram of ground-state total spin of the spinor Fermi gas as a function of its even and odd-wave coupling constants.

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I. INTRODUCTION

The spin-statistics theorem, according to which identical particles with integer spin are bosons whereas those with half-integer spin are fermions, breaks down if the particles are confined to one or two dimensions. Realization of this fact had its origin over forty years ago when it was shown [1, 2] that the many-body problem of hard-sphere bosons in one dimension can be mapped exactly onto that of an ideal Fermi gas, so that many properties of such Bose systems are Fermi-like. It is now known that this “Fermi-Bose duality” is a very general property of identical particles in 1D, not restricted to the hard-sphere model, and relating strongly interacting bosons to weakly-interacting fermions and vice versa. In recent years this esoteric subject has become highly relevant through experiments on ultracold atomic vapors in atom waveguides [3–11]. An understanding of their properties is important for atom interferometry [12, 13] and integrated atom optics [11, 14, 15], which are potentially important for development of ultrasensitive detectors of accelerations and gravitational anomalies.

When an ultracold atomic vapor is placed into an atom waveguide with sufficiently tight transverse confinement, its two-body scattering properties are strongly modified and short-range correlations are greatly enhanced. This occurs in a regime of low temperatures and densities

where both the chemical potential μ and the thermal energy $k_B T$ are less than the transverse oscillator level spacing $\hbar\omega_\perp$, so transverse oscillator modes are frozen and the dynamics is described by a one-dimensional (1D) Hamiltonian with zero-range interactions [16, 17]. This 1D regime has already been reached experimentally [8, 18, 19], as has a regime with $\mu < \hbar\omega_\perp$ but $k_B T > \hbar\omega_\perp$ [6, 10]. We assume herein that both $\mu < \hbar\omega_\perp$ and $k_B T < \hbar\omega_\perp$ as in [8, 18, 19]. Nevertheless, *virtually* excited transverse modes renormalize the effective 1D coupling constant via a confinement-induced resonance, as first shown for bosons [16, 20] and recently for spin-polarized fermionic vapors [21]. At the very low densities of ultracold atomic vapors, the 3D interatomic interactions are usually adequately described by the s-wave scattering length and a corresponding 3D zero-range pseudopotential, the input for the original derivation [16] of an effective 1D interaction between waveguide-confined spinless bosonic atoms. A more detailed and comprehensive theory has been developed recently [22]. In the simplest case of spinless bosons the resultant effective 1D interaction is of the form $g_{1D}^B \delta(z)$ where g_{1D}^B is an explicit and nontrivial function [16, 20, 22] of the 3D s-wave scattering length, $z = z_1 - z_2$, and z_1 and z_2 are 1D coordinates of the interacting atoms measured along the longitudinal axis of the waveguide. This derivation [16, 22] will be described in Sec. II A, and in Sec. II B the derivation of effective 1D pseudopotentials will be presented, for application not only to the spinless Bose gas but also to the case of spinor Fermi and Bose gases, for which the definition of pseudopotentials is much more delicate due to wave function discontinuities induced in the zero-

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range limit by 1D odd-wave interactions derived from 3D p-wave scattering. In the spatially uniform case (no longitudinal trapping potential) the zero-temperature properties of an N -atom spinless Bose gas are determined by a dimensionless coupling constant $\gamma_B = mg_{1D}^B/n\hbar^2$ [16, 17, 23] where m is the atomic mass and $n = N/L$ is the 1D number density, L being the length of the periodic box. The exact N -boson ground state was found in the spatially uniform case by the Bethe Ansatz method in a famous paper of Lieb and Liniger (LL) [23], and spawned later development of a powerful and more general approach [24]. An exact solution in the presence of longitudinal trapping is not known, but is well approximated by a local equilibrium approach [25].

Since the density is in the denominator, at sufficiently low densities one enters the regime of strong interactions and strong short-range correlations where effective-field approaches fail, exactly the opposite of the situation in 3D where the density is in the numerator of the dimensionless coupling constant. For $\gamma_B \gg 1$ the scattering reduces to specular reflection (negligible transmission coefficient) and the Hamiltonian reduces to that of impenetrable point bosons, whose exact N -particle ground state was found in 1960 by an exact mapping to the *ideal* Bose gas [1], leading to “fermionization” of many properties of the Bose gas and related to breakdown of the symmetrization postulate and the spin-statistics theorem [2]. This Fermi-Bose mapping method and its application to determination of exact N -atom ground states will be described in Sec. III A. In Sec. III B a very powerful generalization of this mapping to arbitrary coupling strength [26] will be described and its application to determination of the ground state of the magnetically trapped, spin-aligned Fermi gas will be reviewed, and in Sec. III C application of two further generalizations of the mapping to determination of the ground state of the optically trapped spinor Fermi gas will be described.

II. ATOMIC SCATTERING AND 1D INTERACTIONS IN TIGHT WAVEGUIDES

A. Scattering theory in tight waveguides

1. Formulation of the scattering problem

We begin from the Hamiltonian for two atoms under transverse harmonic confinement and subject to an arbitrary interaction potential

$$\hat{H}_2 = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + \frac{1}{2}m_1\omega_\perp^2\mathbf{r}_{1\perp}^2 + \frac{1}{2}m_2\omega_\perp^2\mathbf{r}_{2\perp}^2 + V(\mathbf{r}_1 - \mathbf{r}_2) \quad (1)$$

where m_1 and m_2 are the atomic masses, ω_\perp is the transverse trap frequency, and ∇_i^2 and $\mathbf{r}_{i\perp}$ are the Laplacian and radial coordinate of the i^{th} atom, respectively. This Hamiltonian is separable in relative and center-of-mass coordinates $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$,

$M = m_1 + m_2$ being the total mass, yielding $\hat{H}_2 = \hat{H}_{rel} + \hat{H}_{COM}$, where

$$\hat{H}_{rel} = -\frac{\hbar^2}{2\mu}\nabla_{\mathbf{r}}^2 + \frac{1}{2}\mu\omega_\perp^2\mathbf{r}_\perp^2 + V(\mathbf{r}), \quad (2)$$

and

$$\hat{H}_{COM} = -\frac{\hbar^2}{2M}\nabla_{\mathbf{R}}^2 + \frac{1}{2}M\omega_\perp^2\mathbf{R}_\perp^2, \quad (3)$$

where $\mu = m_1m_2/(m_1 + m_2)$ is the reduced mass, and \mathbf{r}_\perp and \mathbf{R}_\perp are the relative and center-of-mass radial coordinates, respectively. The center-of-mass Hamiltonian is that of a simple harmonic oscillator whose solution is known, hence we focus only on the relative motion of the two particles. This reduces the problem to a single particle of mass μ , subject to transverse harmonic confinement, which is scattered by an external potential $V(\mathbf{r})$. The central equation which must be solved is therefore Schrödinger’s equation for the state of relative motion of the two atoms

$$[E - \hat{H} - \hat{V}]|\psi(E)\rangle = 0 \quad (4)$$

where $\hat{H} = \hat{H}_{rel}$ and \hat{V} is the interatomic potential. Determining the eigenstates of this Hamiltonian, in particular within the s-wave scattering approximation for $V(\mathbf{r})$, is the central goal of this section.

2. Definitions and theorems

In this subsection we briefly review the T-matrix formulation of scattering theory, which provides a convenient framework for approaching the present problem. Let us first introduce the retarded Green’s function for a system with the Hamiltonian \hat{H} and energy E

$$\hat{G}_{\hat{H}}(E) = \lim_{\epsilon \rightarrow 0^+} (E + i\epsilon - \hat{H})^{-1}. \quad (5)$$

We then define the T-matrix at energy E of the scatter \hat{V} in the presence of the background Hamiltonian \hat{H} in the usual manner as

$$\begin{aligned} \hat{T}_{\hat{H},\hat{V}}(E) &= [1 - \hat{V}\hat{G}_{\hat{H}}(E)]^{-1}\hat{V} \\ &= \sum_{n=0}^{\infty} [\hat{V}\hat{G}_{\hat{H}}(E)]^n \hat{V}, \end{aligned} \quad (6)$$

the summation form being valid provided that there are no difficulties with convergence.

Two relations on which we will rely heavily are the Lippman-Schwinger relation

$$\hat{G}_{\hat{H}+\hat{V}}(E) = \hat{G}_{\hat{H}}(E) + \hat{G}_{\hat{H}}(E)\hat{T}_{\hat{H},\hat{V}}(E)\hat{G}_{\hat{H}}(E), \quad (7)$$

which relates the full Green’s function of the system $\hat{H} + \hat{V}$ to the unperturbed Green’s function $\hat{G}_{\hat{H}}(E)$ and the

T-matrix, and the Lupu-Sax formula [27]

$$\hat{T}_{\hat{H},\hat{V}}(E) = \left[1 - \hat{T}_{\hat{H}',\hat{V}}(E) \left[\hat{G}_{\hat{H}}(E) - \hat{G}_{\hat{H}'}(E)\right]\right]^{-1} \hat{T}_{\hat{H}',\hat{V}}(E), \quad (8)$$

which relates the T-matrix of the scatter \hat{V} in the background Hamiltonian \hat{H} to the T-matrix for the same scatter but in a different background Hamiltonian \hat{H}' .

3. Scattering theory

In the continuous part of the spectrum of the total Hamiltonian $\hat{H} + \hat{V}$ its eigenstates can be expressed as a sum of an incident and a scattered wave according to

$$|\psi(E)\rangle = |\psi_0(E)\rangle + |\psi_s(E)\rangle, \quad (9)$$

where $|\psi_0(E)\rangle$, the “incident” state vector, satisfies

$$\hat{G}_{\hat{H}}^{-1}(E)|\psi_0(E)\rangle = 0, \quad (10)$$

we can then express the Schrödinger equation for the total system as

$$\left[\hat{G}_{\hat{H}}^{-1}(E) - \hat{V}\right] (|\psi_0(E)\rangle + |\psi_s(E)\rangle) = 0. \quad (11)$$

This equation is readily solved for the scattered wave in terms of the unperturbed Green’s function and the T-matrix, yielding

$$|\psi_s(E)\rangle = \hat{G}_{\hat{H}}(E)\hat{T}_{\hat{H},\hat{V}}(E)|\psi_0(E)\rangle, \quad (12)$$

which will serve as the basis for our treatment of the present scattering problem.

4. s-wave scattering regime: the reference T-matrix approach

At first glance it may seem that finding the T-matrix is no easier than a direct solving of the Schrödinger equation (4). We will demonstrate, however, that the T-matrix formulation allows for a self-consistent description of the low-energy part of the spectrum that uses the *free-space* low-energy scattering properties of the interaction potential as the *only* input. In addition the low-energy (s-wave) limit is isolated to a single well-defined approximation without requiring the ad-hoc introduction of regularization via a pseudo-potential. In this section we first outline this self-consistent low-energy treatment. We then solve for the T-matrix using the standard Huang-Fermi pseudo-potential, showing that the pseudo-potential reproduces the exact result in this situation.

Let the unperturbed Hamiltonian \hat{H} be a Hamiltonian for a single nonrelativistic particle in presence of a trapping potential U :

$$\langle \mathbf{r} | \hat{H} | \psi \rangle = \left[-\frac{\hbar^2 \nabla_{\mathbf{r}}^2}{2\mu} + U(\mathbf{r}) \right] \langle \mathbf{r} | \psi \rangle, \quad (13)$$

Assume also that the particle is ‘perturbed’ by a scatterer given by

$$\langle \mathbf{r} | \hat{V} | \psi \rangle = V(\mathbf{r}) \langle \mathbf{r} | \psi \rangle \quad (14)$$

localized around $\mathbf{r} = \mathbf{0}$. In what follows we will derive a *low-energy approximation* for the T-matrix of the scatterer V in presence of \hat{H} . It is important to note that, by definition, the T-matrix acts only on eigenstates of the unperturbed Hamiltonian, which we can safely assume to be regular everywhere. (This is of course a constraint on the properties of the unperturbed Hamiltonian.) In this case the zero-range s-wave scattering limit does not require any regularization of the T-matrix. By making use of the Lupu-Sax formula (8), we first derive the correct form of the T-matrix in the low-energy s-wave regime without the introduction of a regularized pseudo-potential. In the following section, however, we will see that the results we obtain are in agreement with the standard Huang-Fermi pseudopotential approach to s-wave scattering.

We begin our derivation by first specifying a “reference” background Hamiltonian \hat{H}' as

$$\langle \mathbf{r} | \hat{H}' | \psi \rangle = \left[-\frac{\hbar^2 \Delta_{\mathbf{r}}}{2\mu} + E \right] \langle \mathbf{r} | \psi \rangle. \quad (15)$$

This Hamiltonian is that of a free particle, but with an explicit energy dependence included so that the eigenstates have zero wavelength at all energies. We note that this reference Hamiltonian agrees with the free-space Hamiltonian in the zero-energy limit. While this Hamiltonian may seem strange, it is a valid reference Hamiltonian which turns out to be useful because the resulting T-matrix is energy independent for any scattering potential. The Green’s function for this Hamiltonian is given by

$$\langle \mathbf{r} | \hat{G}_{\hat{H}'}(E) | \mathbf{r}' \rangle = -\frac{\mu}{2\pi\hbar^2} \frac{1}{|\mathbf{r} - \mathbf{r}'|}, \quad (16)$$

as can be verified by direct substitution into $[E - \hat{H}'] \hat{G}_{\hat{H}'}(E) = \hat{I}$. In turn the T-matrix of the interaction potential V in presence of \hat{H}' is independent of energy and can therefore be expressed as

$$\langle \mathbf{r} | \hat{T}_{\hat{H}',\hat{V}}(E) | \mathbf{r}' \rangle = g D(\mathbf{r}, \mathbf{r}'), \quad (17)$$

where the kernel D is defined as normalized to unity,

$$\int d\mathbf{r} d\mathbf{r}' D(\mathbf{r}, \mathbf{r}') = 1. \quad (18)$$

The normalization coefficient g is then related, through the zero-energy scattering amplitude, to the three-dimensional scattering length a_s according to

$$g = \frac{2\pi\hbar^2 a_s}{\mu}. \quad (19)$$

Imagine that the kernel $D(\mathbf{r}, \mathbf{r}')$ is well localized within some radius R . In perturbative expansions at low energies this kernel only participates in convolutions with slow (as compare to R) functions, in which case it can be approximated by a δ -function,

$$D(\mathbf{r}, \mathbf{r}') \approx \delta(\mathbf{r})\delta(\mathbf{r}'). \quad (20)$$

This straightforward approximation is the key to the s-wave scattering approximation. This effectively replaces the exact reference T-matrix by its long-wavelength limit, so that the reference T-matrix assumes the form

$$\langle \mathbf{r} | \hat{T}_{\hat{H}', \hat{V}}(E) | \mathbf{r}' \rangle \stackrel{k, k' \ll 1/R}{\approx} g \delta(\mathbf{r}) \delta(\mathbf{r}'), \quad (21)$$

which is equivalent to

$$\hat{T}_{\hat{H}', \hat{V}}(E) = g |0\rangle \langle 0|, \quad (22)$$

where $|0\rangle$ is the position eigenstate corresponding to the location of the scatterer. In expression (21) k and k' refer to the wavevectors of any matrices which multiply the T-matrix from the left and right, respectively.

If we now substitute the above expression for the reference T-matrix into the Lupu-Sax formula (8) for the T-matrix under the background Hamiltonian \hat{H} we arrive at

$$\begin{aligned} \hat{T}_{\hat{H}, \hat{V}}(E) &= \sum_{n=0}^{\infty} \left[g |0\rangle \langle 0| \hat{G}_{\hat{H}'}(E) \right]^n g |0\rangle \langle 0| \\ &= \left[1 - g \langle 0 | \hat{G}_{\hat{H}}(E) | 0 \rangle + g \langle 0 | \hat{G}_{\hat{H}'}(E) | 0 \rangle \right]^{-1} g |0\rangle \langle 0|. \end{aligned} \quad (23)$$

Making use of Eq. (16), we introduce the function $\chi(\mathcal{E})$, defined as

$$\chi(E) = \lim_{\mathbf{r} \rightarrow 0} \left[\langle \mathbf{r} | \hat{G}_{\hat{H}}(E) | 0 \rangle + \frac{\mu}{2\pi\hbar^2 |\mathbf{r}|} \right], \quad (24)$$

from which we obtain the following simple expression for the T-matrix of the scatterer \hat{V} in presence of the trap:

$$\langle \mathbf{r} | \hat{T}_{\hat{H}, \hat{V}}(E) | \psi \rangle \stackrel{E \ll \hbar^2/\mu R^2}{\approx} \frac{g \delta(\mathbf{r})}{1 - g\chi(E)} \langle \mathbf{r} | \psi \rangle. \quad (25)$$

From comparing the equations the free-space and bound Green's functions obey one can show that the singularity in bound Green's function is the same as that in the free-space Green's function. Hence, $\chi(E)$ is the value of the regular part of the bound Green's function at the origin.

For the case of transverse harmonic confinement function $\chi(E)$ has been explicitly computed in [22]. It reads

$$\chi(E) = -\frac{\mu}{2\pi a_{\perp}} \zeta(1/2, -(\frac{E}{2\hbar\omega_{\perp}} - \frac{1}{2})), \quad (26)$$

where $\zeta(s, \alpha)$ is the generalized Riemann zeta function described in the mathematical literature [28]:

$$\begin{aligned} \zeta(s, \alpha) &= \lim_{N \rightarrow \infty} \left[\left(\sum_{n=0}^N \frac{1}{(n+\alpha)^s} \right) - \frac{1}{1-s} \frac{1}{(N+\alpha)^{s-1}} \right] \\ \text{Re}(s) &> 0, \quad -2\pi < \arg(n+\alpha) \leq 0. \end{aligned} \quad (27)$$

Note that no established convention for choosing the branch of the irrational power functions exist: the choice above is just the most suitable for the needs of this paper.

5. Effective one-dimensional interaction potential for waveguide-confined spinless bosons

By a direct substitution to the equation for the Green's function of the relative motion of two particles in a waveguide

$$(E - \hat{H} + i\epsilon) \langle \mathbf{r} | \hat{G}_{\hat{H}}(E) | 0 \rangle = \delta(\mathbf{r}) \quad (28)$$

it is easy to show that the Green's function can be decomposed to a sum over the transverse modes in the following way:

$$\begin{aligned} \langle \mathbf{r} | \hat{G}_{\hat{H}}(E) | 0 \rangle &= \sum_{n=0}^{\infty} \langle z | \hat{G}_{1D}(E - \hbar\omega_{\perp}(2n+1)) | 0 \rangle \phi_n(\boldsymbol{\rho}) \phi_n^*(0), \end{aligned} \quad (29)$$

where $\boldsymbol{\rho} = x\mathbf{e}_x + y\mathbf{e}_y$, $\phi_n(\boldsymbol{\rho})$ are the zero-angular-momentum eigenstates of the transverse oscillator, and $\hat{G}_{1D}(E_{1D})$ is the Green's function of a free one-dimensional particle:

$$\begin{aligned} (E_{1D} + \frac{\hbar^2}{2\mu} \frac{\partial^2}{\partial z^2} + i\epsilon) \langle z | \hat{G}_{1D}(E - \hbar\omega_{\perp}(2n+1)) | 0 \rangle &= \delta(z) \end{aligned} \quad (30)$$

assume now that our energy belongs to the single mode window:

$$\hbar\omega_{\perp} \leq E < 3\hbar\omega_{\perp}. \quad (31)$$

In this case all the $n \neq 0$ terms in the expansion (30) exponentially decay at large z . Accordingly the solution of the scattering problem (9), (12) now resembles a solution of a *one-dimensional* scattering problem:

$$\begin{aligned} \psi(\mathbf{r}) \stackrel{|z| \rightarrow \infty}{\equiv} & \{ \psi_{0,1D}(z) + \\ & \langle z | \hat{G}_{1D}(E_{1D}(E)) | 0 \rangle | \phi_0(0) |^2 \tau_{3D}(E) \psi_{0,1D}(0) \} \phi_0(\boldsymbol{\rho}), \end{aligned} \quad (32)$$

where $\tau_{3D}(E)$ is the strength of the three-dimensional T-matrix, i.e.,

$$\tau_{3D}(E) = \frac{g \delta(\mathbf{r})}{1 - g\chi(E)}, \quad (33)$$

and $E_{1D}(E) = E - \hbar\omega_\perp$. Comparing the expression (33) and the general form of a scattering solution (9), (12) it is natural to interpret the expression in the braces in the l.h.s. of (33) as a solution of a *one-dimensional* scattering problem, subject to a scattering potential whose (one-dimensional) T-matrix reads

$$\langle z | \hat{T}_{1D}(E_{1D}) | 0 \rangle = \tau_{1D}(E_{1D}) \delta(z) \quad (34)$$

with

$$\tau_{1D}(E_{1D}) = |\phi_0(0)|^2 \tau_{3D}(\hbar\omega_\perp + E_{1D}). \quad (35)$$

For low energies $E_{1D} \ll \hbar\omega_\perp$ the strength of the one-dimensional T-matrix is approximately

$$\tau_{1D}(E_{1D}) \stackrel{E_{1D} \ll \hbar\omega_\perp}{\approx} -\frac{\hbar^2}{\mu} \frac{1}{-\frac{a_s^2}{2a_s} \left[1 + \frac{a_s}{a_\perp} \zeta(1/2) \right] - \frac{i}{k}} \quad (36)$$

The relative error of this approximation scales as $\mathcal{O}(k^3)$.

One can now attempt to introduce an effective one-dimensional scatterer whose T-matrix is close or equal to the one given above (34), (36). A straightforward calculation shows that the T-matrix of a one-dimensional delta-potential

$$\hat{V}_{1D} = g_{1D}^B \delta(z) \quad (37)$$

has a form

$$\tau_{1D,\delta}(E_{1D}) = -\frac{\hbar^2}{\mu} \frac{1}{\frac{a_{1D}^B}{a_{1D}^B} - \frac{i}{k}}, \quad (38)$$

where the one-dimensional scattering length a_{1D}^B is related to the potential strength by

$$g_{1D}^B = -\frac{\hbar^2}{\mu a_{1D}^B}. \quad (39)$$

Comparison of the T-matrices (36) and (38) leads to a conclusion that the waveguide scattering T-matrix (36) can be exactly reproduced by a delta-potential (37) of a scattering length

$$a_{1D}^B = -\frac{a_\perp^2}{2a_s} \left[1 + \frac{a_s}{a_\perp} \zeta(1/2) \right], \quad (40)$$

where $\zeta(x)$ is the Riemann zeta function and $\zeta(1/2) = -1.4603\dots$

The above expression reproduces the result of [16] where the effective one-dimensional potential has been obtained via a straightforward solution of the scattering problem. Notice that at $a_s = |\zeta(1/2)|^{-1} a_\perp$ the coupling constant diverges, signifying the so-called “confinement induced resonance” (CIR). The significance of this resonance has been confirmed in *ab initio* two-body numerical calculations with finite-range realistic interatomic potentials [20] and in many-body Monte-Carlo simulations [29].

Further extensions of the zero-range model for the effective one-dimensional scatterer can be envisioned. For

example, if one chooses to reproduce the scattering properties (more precisely the denominator of the one-dimensional T-matrix (35)) with a relative error of $\mathcal{O}(k^5)$, the one-dimensional delta-potential can be replaced by a rectangular potential of a finite width $2l$ and height/depth v_0 . In the case of repulsive interaction the model potential is a rectangular barrier, and for attractive interactions it is a rectangular well. In the limit of $a_s \ll a_\perp$ their half-widths l are given by

$$l = \frac{\sqrt{\zeta(3/2)}}{2} \sqrt{a_s a_\perp} = 0.8081 \sqrt{a_s a_\perp} \quad \text{for } g_{1D}^B > 0, \\ l = \frac{(\zeta(3/2)/2)^{2/3}}{18} a_s = .0664 |a_s| \quad \text{for } g_{1D}^B < 0,$$

and in both cases the strength v_0 of the potential is

$$v_0 = |g_{1D}^B/2l|. \quad (41)$$

6. Remarks on p-wave scattering

Attempts to carry through a program analogous to the above for the case of p-wave scattering between polarized fermions meet numerous (hopefully technical) obstacles: Most of the limiting procedures become mutually nonuniformly convergent and no clear way to identify a correct order is visible. In any case the closest candidate for the p-wave analog of the free-space three-dimensional T-matrix is the pseudopotential introduced in [30]

$$\langle \chi | \hat{T}_{\hat{H}', \hat{V}}(E) | \psi \rangle = \frac{27\pi\hbar^2 V_p}{\mu} \nabla \chi^*(0) \cdot \nabla \psi(0) \quad (42)$$

that can be shown to reproduce correctly the low-energy behavior of the p-wave scattering amplitude. Here V_p is the p-wave scattering volume, that defines the low-energy behavior of the p-wave scattering phase via $V_p = -\lim_{k \rightarrow 0} \tan \delta_p(k)/k^3$ [31].

An elegant way around these difficulties has been found recently by Granger and Blume [21], who used a K-matrix technique that does not explicitly involve any zero-range objects. The analysis of polarized Fermi gases presented below is heavily based on the Granger and Blume findings.

B. 1D pseudopotentials

1. Spinless bosons

This is the simplest case. The 1D scattering length a_{1D}^B is defined in terms of the ratio of derivative $\psi_B'(z)$ and value $\psi_B(z)$ of the relative wave function just outside the range z_0 of the interaction:

$$\psi_B'(z_0) = -\psi_B'(-z_0) = -(a_{1D}^B - z_0)^{-1} \psi_B(\pm z_0) \quad , \quad (43)$$

which is equivalent in the zero-range limit $z_0 \rightarrow 0+$ to the familiar LL *contact condition* [23, 24]

$$\psi_B'(0+) = -\psi_B'(0-) = \frac{\mu g_{1D}^B}{\hbar^2} \psi_B(0\pm) \quad (44)$$

for the delta function interaction $g_{1D}^B \delta(z)$ provided that the scattering length and coupling constant are related by $g_{1D}^B = -\hbar^2/\mu a_{1D}^B$ where μ is the effective mass $m/2$. It follows from the expression for a_{1D}^B derived in [16, 22] and Eq.(40) herein that

$$g_{1D}^B = 2a_s \hbar \omega_\perp \left[1 - \frac{a_s}{a_\perp} |\zeta(1/2)| \right]^{-1} \quad (45)$$

implying the existence of a confinement-induced resonance CIR [16, 20] of the coupling constant as a_s is tuned via a 3D Feshbach resonance [32] past the resonance point $a_s/a_\perp = |\zeta(1/2)|^{-1} = 0.6848\dots$. Hence the whole range of 1D coupling constants from $-\infty$ to $+\infty$ is experimentally achievable by tuning a_s over a narrow range in the neighborhood of the 1D resonance. It was shown recently [20] that this is a 1D Feshbach resonance between ground and excited transverse vibrational manifolds. It was shown in [20] and in Sec. II A 5 herein that at low longitudinal energies $ka_\perp \ll 1$ the 1D scattering amplitude generated by the interaction $g_{1D}^B \delta(z)$ reproduces the exact 3D scattering amplitude in the waveguide to within a relative error $\mathcal{O}(k^3)$.

2. Spin-aligned fermions

Consider next a magnetically trapped, spin-aligned atomic vapor of spin- $\frac{1}{2}$ fermionic atoms in a tight waveguide. The N -fermion spin wave function is magnetically frozen in the configuration $\uparrow_1 \cdots \uparrow_N$, so the space-spin wave function must be *spatially* antisymmetric, s-wave scattering is forbidden, and the leading interaction effects at low energies are determined by the 3D p-wave scattering amplitude. Such p-wave interactions are usually negligible at the low densities of ultracold atomic vapors, but they can be greatly enhanced by p-wave Feshbach resonances [33]. Granger and Blume derived an effective one-dimensional K-matrix for the corresponding two-fermion problem [21] in a tight waveguide. In the low-energy [34] domain the K-matrix can be reproduced, with a relative error $\mathcal{O}(k^3)$, by the contact condition [21, 35]

$$\psi_F(0+) = -\psi_F(0-) = -a_{1D}^F \psi_F'(0\pm) \quad (46)$$

where

$$a_{1D}^F = \frac{6V_p}{a_\perp^2} [1 + 12(V_p/a_\perp^3) |\zeta(-1/2, 1)|]^{-1} \quad (47)$$

is the odd-wave one-dimensional scattering length, $V_p = a_p^3 = -\lim_{k \rightarrow 0} \tan \delta_p(k)/k^3$ is the p-wave “scattering volume” [31], a_p is the p-wave scattering length, and $\zeta(-1/2, 1) = -\zeta(3/2)/4\pi = -0.2079\dots$ is the Hurwitz zeta function evaluated at $(-1/2, 1)$ [36]. The expression (47) has a resonance at a *negative* critical value $V_p^{crit}/a_\perp^3 = -0.4009\dots$. Note that $a_{1D}^F \rightarrow -\infty$ as V_p approaches this critical value, implying that the exterior wave function (i.e., outside the interaction region

$|z| < z_0 \rightarrow 0$) satisfies the *free-particle* Schrödinger equation. This is the opposite of the bosonic case, where it follows from Eq. (40) that $a_{1D}^B \rightarrow 0$ at resonance.

In accordance with (46), the low-energy fermionic wavefunctions, Eq. (20) of [21], are discontinuous at contact, but left and right limits of their derivatives coincide. V_p is tunable via a 3D Feshbach resonance [33], allowing experimental realization of all values of a_{1D}^F from $-\infty$ to $+\infty$. It will be shown that in this fermionic case the effective 1D coupling constant is $g_{1D}^F = -\hbar^2 a_{1D}^F/\mu$, which can be compared and contrasted with the previously defined bosonic 1D coupling constant $g_{1D}^B = -\hbar^2/\mu a_{1D}^B$. The dimensionless fermionic coupling constant is $\gamma_F = mg_{1D}^F n/\hbar^2$. Note that the density n is in the numerator, whereas it is in the denominator of the bosonic analog $\gamma_B = mg_{1D}^B/n\hbar^2$. For $\gamma_F \gg 1$ one has a “fermionic TG gas” [35], a fermionic analog of the impenetrable Bose gas, called the “Tonks-Girardeau” (TG) gas in recent literature [14, 18, 25, 29, 37–48]. As previously noted, $a_{1D}^F \rightarrow -\infty$ in the fermionic TG limit, implying an interaction-free exterior wave function. It will be shown in Sec. III B that in this limit the Fermi gas maps to an *ideal Bose* gas, providing a physical explanation of the interaction-free nature of the exterior wave function. This can be compared and contrasted with the bosonic TG gas, which maps to an *ideal Fermi* gas.

Although a discontinuity in the derivative is a well-known consequence of the zero-range delta function pseudopotential and plays a crucial role in the solution of the Lieb-Liniger model [23], discontinuities of ψ itself have received little attention, although they have been discussed previously by Cheon and Shigehara [26] and are implicit in the recent work of Granger and Blume [21]. For a fermionic wave function ψ_F the discontinuity $2\psi_F(0+)$ is a trivial consequence of antisymmetry together with the fact that a nonzero odd-wave scattering length cannot be obtained in the limit $z_0 \rightarrow 0$ unless $\psi_F(0\pm) \neq 0$. These discontinuities are rounded off when $z_0 > 0$, since the interior wave function interpolates smoothly between the values at $z = -z_0$ and $z = z_0$. This is illustrated in Fig. 1 for the special case of the fermionic TG gas, the limit $\gamma_F \gg 1$ [35]. The potential is chosen to be a square *well* because we will find later that stability of the ground state against collapse requires that the corresponding effective zero-range interaction ($z_0 \rightarrow 0+$) be *negative* definite, which can be shown to be the case when $g_{1D}^F > 0$ and hence both $V_p < 0$ and $a_{1D}^F < 0$. The energy is taken as zero so the exterior solution is $\text{sgn}(z) = \pm 1$; an interior solution fitting smoothly onto this is $\sin(\kappa z)$ with $\kappa = \sqrt{2\mu V_0/\hbar^2} = \pi/2z_0$, the critical value where the last bound state passes into the continuum, a zero-energy resonance. A fermionic contact condition with a finite scattering length can be obtained in the limit $z_0 \rightarrow 0$ if κ scales with the width z_0 as $\kappa = (\pi/2z_0)[1 + (2/\pi)^2(z_0/a_{1D}^F)]$.

Following the bosonic case, where the δ -interaction can be introduced naturally to cancel the δ -functions resulting from double differentiation of functions with discontinuities,

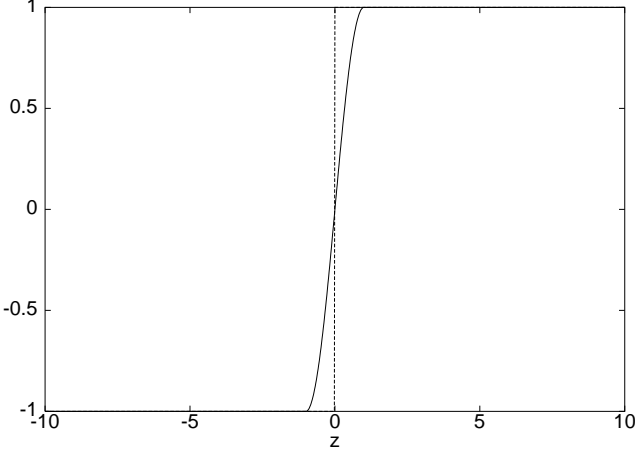


FIG. 1: $N = 2$ untrapped fermionic TG gas ground state (dashed line) compared with zero-energy scattering solution for a square well with range z_0 and depth V_0 corresponding to the boundary between no bound state and one bound state, a zero energy resonance (solid line), as function of relative coordinate z . Units are such that $z_0 = 1$.

tinuous derivatives, in the case of fermions whose wave function is discontinuous it is tempting to introduce δ' interactions. However, δ' -functions and second derivatives are ill-defined if used in a convolution with discontinuous functions. This difficulty is resolved by realizing that (as will be shown), the δ' is associated with the *interior* wave function ($-z_0 < z < z_0$) whereas the contact condition (46) refers to the $z_0 \rightarrow 0+$ limit of the wave function at $z = \pm z_0 \neq 0$, just *outside* the range of the interaction potential. Take the Hamiltonian to be $\hat{H}_{1D}^F = -(\hbar^2/2\mu)\partial_z^2 + \hat{v}_{1D}^F$ where \hat{v}_{1D}^F is a pseudopotential operator to be determined. ∂_z^2 is non-singular for $z \neq 0$, but at the origin there are singular contributions. The first derivative is $\partial_z \psi_F(z) = \psi'_F(z \neq 0) + [\psi_F(0+) - \psi_F(0-)]\delta(z)$. The second derivative then has an additional singular contribution from differentiation of the delta function:

$$\partial_z^2 \psi_F(z) = \psi''_F(z \neq 0) + [\psi_F(0+) - \psi_F(0-)]\delta'(z) \quad (48)$$

In the second term on the right, $\delta'(z)$ is associated with the interior wave function, as we will eventually verify explicitly from the $z \rightarrow 0+$ limit of a finite-range interaction, whereas its prefactor arises from the contact condition just outside the range of the interaction. Define a linear regularizing operator $\hat{\partial}_\pm$ by

$$\hat{\partial}_\pm \psi_F(z) = (1/2)[\psi'_F(0+) + \psi'_F(0-)] \quad (49)$$

Then the desired pseudopotential is $\hat{v}_{1D}^F = g_{1D}^F \delta'(z) \hat{\partial}_\pm$. It satisfies a convenient projection property $\hat{v}_{1D}^F \psi_e(z) = 0$ where ψ_e is any even wave function. Although ψ_F is always purely odd (antisymmetric) for the spin-aligned Fermi gas, this projection property is useful for the spinor (spin-free) Fermi gas to be discussed in Sec. II B 3, where the spatial dependence has, in general, both even and odd contributions. This is the reason for the general

form $(1/2)[\psi'_F(0+) + \psi'_F(0-)]$ of the factor occurring in Eq. (49), even though it could be simplified to $\psi'_F(0+)$ or $\psi'_F(0-)$ when acting on ψ_F , for which $\psi'_F(0+) = \psi'_F(0-)$ as a consequence of antisymmetry. It is easy to show that terms in $\delta'(z)$ cancel from $\hat{H}_{1D}^F \psi_F$ as a consequence of the contact condition (46) provided that $g_{1D}^F = -\hbar^2 a_{1D}^F / \mu$.

The physical significance is clarified by starting from the same square well already discussed in connection with Fig. 1, i.e., $v(z) = -V_0$ when $-z_0 < z < z_0$ and zero when $|z| > z_0$. The antisymmetric solution ψ_F of the zero-energy scattering equation $[(-\hbar^2/2\mu)\partial_z^2 + v(z)]\psi_F(z) = 0$ inside the well is $\sin(\kappa z)$ with $\kappa = \sqrt{2\mu V_0/\hbar^2}$. The scattering length a_{1D}^F is defined by $\psi_F(z_0) = -\psi_F(-z_0) = -(a_{1D}^F - z_0)\psi'_F(\pm z_0)$ which is satisfied in the limit $z_0 \rightarrow 0+$ if V_0 scales with z_0 as $\kappa = (\pi/2z_0)[1 + (2/\pi)^2(z_0/a_{1D}^F)]$. In that limit the boundary conditions reduce to Eq. (46). Inside the well the kinetic and potential energy terms are $-(\hbar^2/2\mu)\partial_z^2 \psi_F(z) = -(\hbar^2 \kappa^2/2\mu) \sin(\kappa z)$ and $v(z)\psi_F(z) = -V_0 \sin(\kappa z)$. For $|z| < z_0$, $\cos(\kappa z)$ is proportional to a representation of $\delta(z)$ as $z_0 \rightarrow 0$, since $\int_{-z_0}^{z_0} \cos(\kappa z) f(z) dz \rightarrow f(0) \int_{-z_0}^{z_0} \cos(\kappa z) dz = f(0) 2\kappa^{-1} \sin(\kappa z_0) \rightarrow 2z_0 f(0)$. Then its derivative $-(\kappa/2z_0) \sin(\kappa z)$ is a representation of $\delta'(z)$. Noting that $\kappa z_0 \rightarrow \pi/2$ as $z_0 \rightarrow 0$ we have $-(\hbar^2/2\mu)\partial_z^2 \psi_F(z) = -(\hbar^2 \kappa^2/2\mu) \sin(\kappa z) \rightarrow (\pi \hbar^2/2\mu) \delta'(z)$ which agrees with the kinetic energy term $-(\hbar^2/2\mu)[\psi_F(0+) - \psi_F(0-)]\delta'(z)$ since $\psi_F(0+)$ and $\psi_F(0-)$ are to be interpreted as $\psi_F(z_0)$ and $\psi_F(-z_0)$ as $z_0 \rightarrow 0+$. Next consider the potential energy term inside the well as $z_0 \rightarrow 0+$: $-V_0 \sin(\kappa z) \rightarrow -V_0(-2z_0/\kappa)\delta'(z) \rightarrow (\pi \hbar^2/2\mu)\delta'(z)$. Comparing this with $\hat{v}_{1D}^F \psi_F(z)$, using the expression for g_{1D}^F , noting that $\psi'_F(0\pm)$ are to be interpreted as $\psi'_F(\pm z_0)$ for $z_0 \rightarrow 0$, one finds that the two expressions for the potential energy term agree in that limit. It is clear from this derivation that the $\delta'(z)$ in the effective 1D Hamiltonian can be interpreted as the “ghost of the vanished interior wave function”, which plays a crucial role in the odd-wave interaction even in the limit $z_0 \rightarrow 0$.

3. Spin-free fermions

In this section we assume that the spinor Fermi gas is optically trapped, so the spins are unconstrained. This case is more complicated than the spin-aligned Fermi gas even in the absence of explicit spin-spin interactions or external spin-dependent potentials, because the requirement of antisymmetry under combined space-spin exchanges $(z_i, \sigma_i) \leftrightarrow (z_j, \sigma_j)$ induces implicit space-spin coupling leading to nontrivial spin dependence of the wave functions. Here each spin z -component argument σ_i takes on the values \uparrow or \downarrow , or equivalently $\pm \frac{1}{2}$. Consider 3D two-body scattering in the waveguide. There are both s-wave scattering states, which are space symmetric and spin antisymmetric with spin eigenfunctions of singlet

form $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$, and p-wave scattering states, which are space antisymmetric and spin symmetric with spin eigenfunctions of triplet form $\uparrow\uparrow$ or $\downarrow\downarrow$ or $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$. Assume that the Hamiltonian does not depend on spin. Then the spin dependence of fermionic wave functions ψ_F need not be indicated explicitly and they can be written as the sum of spatially even and odd parts ψ_e and ψ_o . The odd part decomposes further into three components going with the three triplet spin eigenfunctions, but one need not complicate the notation at this point since the Hamiltonian acts in the same way on each of them. The effective 1D interactions are determined by 1D scattering lengths a_{1D}^e for spatially even waves $\psi_e(z) = \psi_e(-z)$ related to 3D s-wave scattering and spatially odd waves $\psi_o(z) = -\psi_o(-z)$ related to 3D p-wave scattering. The contact condition for 1D even-wave scattering is the same as the previously-given one (44) with ψ_B and a_{1D}^B replaced by ψ_e and a_{1D}^e , and the one for 1D odd-wave scattering is $\psi_o(z_0) = -\psi_o(-z_0) = -(a_{1D}^o - z_0)\psi_o'(\pm z_0)$, the $z_0 > 0$ version of Eq. (46), with ψ_F and a_{1D}^F replaced by ψ_o and a_{1D}^o . In the zero-range limit $z_0 \rightarrow 0+$ these can be combined into [35, 49]

$$\begin{aligned}\psi'(0+) - \psi'(0-) &= -(a_{1D}^e)^{-1}[\psi(0+) + \psi(0-)] \\ \psi(0+) - \psi(0-) &= -a_{1D}^o[\psi'(0+) + \psi'(0-)]\end{aligned}\quad (50)$$

where $\psi(z) = \psi_e(z) + \psi_o(z)$. In this section $\psi(z)$ is a fermionic function ψ_F , but we use a more general notation because the same equations apply to a spinor Bose gas. a_{1D}^e and a_{1D}^o are related to the 3D s-wave scattering length a_s and the 3D p-wave scattering volume V_p by Eqs. (40) and (47), with a_{1D}^B replaced by a_{1D}^e and a_{1D}^F replaced by a_{1D}^o . Take the Hamiltonian to be

$$\hat{H}_{1D} = -(\hbar^2/2\mu)\partial_z^2 + \hat{v}_{1D}^e + \hat{v}_{1D}^o \quad (51)$$

Here \hat{v}_{1D}^o differs from \hat{v}_{1D}^F of Sec. IIB 2 only by the obvious substitutions, i.e., $\hat{v}_{1D}^o = g_{1D}^o \delta'(z) \hat{\partial}_\pm$ with $\hat{\partial}_\pm$ defined by Eq. (49) with ψ_F replaced by ψ_o . The even-wave pseudopotential is more complicated than the simple delta-function interaction of Sec. IIB 1, because the delta function is ambiguous at the point $z = 0$, where ψ is discontinuous due to the discontinuity in its odd-wave component ψ_o . In order to determine the correct form, start from the first derivative $\partial_z \psi(z) = \psi'(z \neq 0) + [\psi(0+) - \psi(0-)]\delta(z)$ as before. Now the second derivative has *two* singular contributions in addition to the nonsingular term $\psi''(z \neq 0)$, one because in general $\psi'(0+) \neq \psi'(0-)$ and the other from the derivative of the delta function:

$$\begin{aligned}\partial_z^2 \psi(z) &= \psi''(z \neq 0) + [\psi'(0+) - \psi'(0-)]\delta(z) \\ &+ [\psi(0+) - \psi(0-)]\delta'(z).\end{aligned}\quad (52)$$

With proper choice of g_{1D}^o the odd-wave pseudopotential \hat{v}_{1D}^o cancels the $\delta'(z)$ term from the kinetic energy, and we define the even-wave pseudopotential \hat{v}_{1D}^e so as to cancel

the $\delta(z)$ term: $\hat{v}_{1D}^e = g_{1D}^e \hat{\delta}_\pm$ where the linear operator $\hat{\delta}_\pm$ is defined by

$$\hat{\delta}_\pm \psi(z) = (1/2)[\psi(0+) + \psi(0-)]\delta(z) \quad (53)$$

These pseudopotentials satisfy convenient projection properties $\hat{v}_{1D}^e \psi_o = \hat{v}_{1D}^o \psi_e = 0$ on the even and odd parts of ψ , and their matrix elements are $\langle \chi | v_{1D}^e | \psi \rangle = \frac{1}{2} \chi^*(0)[\psi(0+) + \psi(0-)]$ and $\langle \chi | v_{1D}^o | \psi \rangle = -\frac{1}{2} [\chi'(0)]^* [\psi'(0+) + \psi'(0-)]$. They connect only even to even and odd to odd wave functions if we stipulate that $\chi(0) = 0$ [the average of $\chi(0+)$ and $\chi(0-)$] if χ is odd and $\chi'(0) = 0$ [the average of $\chi'(0+)$ and $\chi'(0-)$] if χ is even. In fact, the wave function and its derivative at $z = 0$ refer to the *internal* wave function as modified by the potential, whereas $z = 0+$ and $z = 0-$ refer to the wave function *just outside* the range of the potential, and the above values at $z = 0$ follow from the way the internal wave function interpolates between the contact conditions on the *exterior* wave function. Using (50) one finds that terms in $\delta(z)$ and $\delta'(z)$ cancel from \hat{H}_{1D} if the even and odd-wave coupling constants are related to the scattering lengths by $g_{1D}^e = -\hbar^2/\mu a_{1D}^e$ and $g_{1D}^o = -\hbar^2 a_{1D}^o/\mu$.

III. FERMION-BOSE MAPPING METHODS AND N-ATOM GROUND STATES

In this section we will review the theory of Fermi-Bose mappings relating the exact N -particle energy eigenstates of systems of fermions and bosons in 1D with effective zero-range interactions, and application of these mappings to determination of the N -atom ground states. This will be done first for the original mapping for impenetrable bosons ($\gamma \gg 1$) [1, 2], then for a very powerful generalization to arbitrary values of γ due to Cheon and Shigehara[26], and finishing with a further generalization to the case of spinor Fermi and Bose gases, important for applications to optically trapped fermionic atoms whose spins are unconstrained.

A. Impenetrable bosons (TG limit)

It was already pointed out in the famous paper of Lieb and Liniger on the 1D Bose gas with delta-function interactions [23] and in Secs. I and IIB 2 that the 1D gas of impenetrable point bosons is the limit $\gamma_B \gg 1$ of the LL gas, the “TG limit”. Tonks gave the first treatment of the statistical mechanics of a 1D hard-sphere gas [50], which was restricted to the classical high-temperature regime and provided no information about the extreme quantum limit characteristic of ultracold atomic vapors. The formula for the exact quantum-mechanical ground-state energy of the 1D hard-sphere Bose gas appeared in a paper of Bijl where it is quoted without derivation [51], and a derivation was published by Nagamiya [52]. Then in

1960 one of us [1] and Stachowiak [53] independently re-deriveded the ground-state energy. The Fermi-Bose mapping method was first introduced in [1], although Nagamiya had previously noted [52] that in the “fundamental sector” $z_1 \leq z_2 \leq \dots \leq z_N$ the ground state wave function of a spatially uniform, 1D hard-core Bose gas can be written as an ideal Fermi gas determinant, continuation into other permutation sectors being effected by imposing overall Bose symmetry under all permutations $z_i \leftrightarrow z_j$ in spite of the fermionic *antisymmetry* under permutations of *orbitals* (not coordinates) in the fundamental sector. The mapping theorem is much more general, also holding in the presence of external potentials and/or finite two-particle or many-particle interactions in addition to the hard core interaction [1]. It also applies to the 1D time-dependent many-body Schrödinger equation and has been used to treat some time-dependent interference properties of the 1D hard core Bose gas [54–58].

We now briefly review the mapping theorem. The N -boson Hamiltonian is assumed to have the structure $\hat{H}_{1D} = -(\hbar^2/2m) \sum_{j=1}^N \partial_{z_j}^2 + V(z_1, \dots, z_N)$ where the real, symmetric function V contains all external potentials (e.g., a longitudinal trap potential) as well as any finite interaction potentials *not including* the hard-sphere repulsion, which is instead treated as a constraint on allowed wave functions $\psi_B(z_1, \dots, z_N)$:

$$\psi_B = 0 \quad \text{if} \quad |z_j - z_k| < a, \quad 1 \leq j < k \leq N. \quad (54)$$

Let $\psi_F(z_1, \dots, z_N)$ be a fermionic solution of $\hat{H}_{1D}\psi = E\psi$ which is antisymmetric under all pair exchanges $z_j \leftrightarrow z_k$, hence all permutations. One can consider ψ_F to be either the wave function of a fictitious system of “spinless fermions”, or else that of a system of real, spin-aligned fermions. Define a “unit antisymmetric function” [1]

$$A(z_1, \dots, z_N) = \prod_{1 \leq j < k \leq N} \text{sgn}(z_k - z_j), \quad (55)$$

where $\text{sgn}(z)$ is the algebraic sign of the coordinate difference $z = z_k - z_j$, i.e., it is +1(-1) if $z > 0$ ($z < 0$). For given antisymmetric ψ_F , define a bosonic wave function ψ_B by

$$\psi_B(z_1, \dots, z_N) = A(z_1, \dots, z_N) \psi_F(z_1, \dots, z_N) \quad (56)$$

which defines the Fermi-Bose mapping. ψ_B satisfies the hard core constraint (54) if ψ_F does, is totally symmetric (bosonic) under permutations, obeys the same boundary conditions as ψ_F , and $\hat{H}_{1D}\psi_B = E\psi_B$ follows from $\hat{H}_{1D}\psi_F = E\psi_F$ [1, 2]. In the case of periodic boundary conditions (no trap potential, spatially uniform system) one must add the proviso that the boundary conditions are only preserved under the mapping if N is odd, but the case of even N is accommodated by imposing periodic boundary conditions on ψ_F but *antiperiodic* boundary conditions on ψ_B .

The mapping theorem leads to explicit expressions for all many-body energy eigenstates and eigenvalues under the assumption that the only two-particle interaction is a zero-range hard core repulsion, represented by the $a \rightarrow 0$ limit of the hard-core constraint, the “TG gas”. Such solutions were obtained in Sec. 3 of [1] for periodic boundary conditions and no external potential. At the low densities of ultracold atomic vapors it is usually sufficient to consider this case, although it has been shown by Astrakharchik *et al.* [29] that for a longitudinally trapped LL gas with attractive interaction $g_{1D} < 0$ there is a regime within which the equation of state is well approximated by taking $a > 0$. The exact ground state is also known in this case [1, 52]. Here we limit ourselves to the usual strongly repulsive TG limit $\gamma_B \gg 1$. Since wave functions of “spinless” or spin-aligned fermions are antisymmetric under coordinate exchanges, their wave functions vanish automatically whenever any $z_j = z_k$, the constraint has no effect, and the corresponding fermionic ground state is the ground state of the *ideal* gas of fermions, a Slater determinant of the lowest N single-particle plane-wave orbitals. The exact many body ground state was found [1] to have energy $E_0 = (N - N^{-1})(\pi\hbar n)^2/6m$ where $n = N/L$ is the linear number density, and the wave function was found to be a pair product of Bijl-Jastrow form

$$\psi_{B0} = \text{const.} \prod_{i>j} |\sin[\pi L^{-1}(z_i - z_j)]|, \quad (57)$$

where L is the perimeter of the annular trap. In spite of the very long range of the individual pair correlation factors $|\sin[\pi L^{-1}(z_i - z_j)]|$, the pair distribution function $D(z_{ij})$, the joint probability density that if one particle is found at z_i a second will be found at z_j , was found to be of short range $D(z_{ij}) = 1 - [\sin(\pi n z_{ij})/\pi n z_{ij}]^2$. Clearly, $D(0) = 0$ which reflects the hard core nature of the two-particle interaction. By examination of the excited states the system was found to support propagation of sound with speed $c = \pi\hbar n/m$ [1], and it was shown that this agrees with the thermodynamic formula in terms of the compressibility of the ground state. “Fermionization” holds only for those properties expressible in terms of the configurational probability density $|\psi_{B0}(z_1, z_2, \dots, z_N)|^2$. The momentum distribution depends on the single-particle correlation function $g_1(z) = \langle \hat{\psi}^\dagger(z) \hat{\psi}(0) \rangle$ (reduced single-particle density matrix), which is very different from that of the ideal Fermi gas and very difficult to evaluate. Its eigenfunctions are plane waves e^{ikz} because of translational invariance of the system, and the corresponding eigenvalues define the momentum distribution function $N(k)$, the discrete Fourier transform of $g_1(z)$, the allowed values of k being $k_j = j2\pi/L$ with $j = 0, \pm 1, \pm 2, \dots$. In a classic *tour de force* Lenard found [59, 60] $N(0)$ to be of order \sqrt{N} , large but much less than the $\mathcal{O}(N)$ value required for Bose-Einstein condensation. More generally, $g_1(z)$ was found [61] to be of order $1/\sqrt{k}$ at small k . The corresponding momentum distribution is sharply peaked at low k and

falls like k^{-4} at large k [37, 39, 62], very different from the filled Fermi sea of the ideal Fermi gas, for which $N(k)$ is unity for $|k| < k_F$ and zero for $|k| > k_F$.

The exact ground state of the TG gas is also known in the presence of a longitudinal trap potential $\frac{1}{2}m\omega_{\text{long}}^2 z^2$ [63]. It follows from the mapping theorem that the exact N -boson ground state ψ_{B0} is $\psi_{B0}(z_1, \dots, z_N) = |\psi_{F0}(z_1, \dots, z_N)|$ where ψ_{F0} is the ground state of N spinless fermions with the same Hamiltonian and impenetrability constraint. The fermionic ground state is a Slater determinant of the lowest N single-particle eigenfunctions φ_n of the harmonic oscillator (HO), where $\varphi_n(z) = \text{const.} e^{-Q^2/2} H_n(Q)$ with $H_n(Q)$ the Hermite polynomials and $Q = z/z_{\text{osc}}$, $z_{\text{osc}} = \sqrt{\hbar/m\omega_{\text{long}}}$ being the longitudinal oscillator length. By factoring the Gaussians out of the determinant and carrying out elementary row and column operations, one can cancel all terms in each H_n except the one of highest degree [64], yielding a simple but exact analytical expression of Bijl-Jastrow pair product form for the N -boson ground state:

$$\psi_{B0}(z_1, \dots, z_N) = \text{const.} \left[\prod_{i=1}^N e^{-Q_i^2/2} \right] \prod_{1 \leq j < k \leq N} |z_k - z_j| \quad (58)$$

with $Q_i = z_i/z_{\text{osc}}$. It is interesting to note the strong similarity between this exact 1D N -boson wave function and the famous Laughlin variational wave function of the 2D ground state for the quantized fractional Hall effect [65], as well as the closely-related wave functions for bosons with weak repulsive delta-function interactions in a harmonic trap in 2D found by Smith and Wilkin [66].

Both the single particle density and pair distribution function depend only on the absolute square of the many-body wave function, and since $|\psi_{B0}|^2 = |\psi_{F0}|^2$ they reduce to standard ideal Fermi gas expressions. The single particle density, normalized to N , is $n(z) = \sum_{n=0}^{N-1} |\varphi_n(z)|^2$ and the pair distribution function, normalized to $N(N-1)$, is

$$\begin{aligned} D(z_1, z_2) &= n(z_1)n(z_2) - |\Delta(z_1, z_2)|^2 \\ \Delta(z_1, z_2) &= \sum_{n=0}^{N-1} \varphi_n^*(z_1) \varphi_n(z_2) \quad . \end{aligned} \quad (59)$$

Although the Hermite polynomials have disappeared from the expression (58) for the many-body wave function, they reappear upon integrating $|\psi_{B0}|^2$ over $(N-1)$ coordinates to get the single particle density $n(z)$ and over $(N-2)$ to get the pair distribution function $D(z_1, z_2)$, and these expressions in terms of the HO orbitals φ_n are the most convenient ones for evaluation. Some qualitative features of the pair distribution function are apparent: In the first place it vanishes at contact $z_1 = z_2$, as it must because of impenetrability of the particles. Furthermore, the correlation term $\Delta(z_1, z_2)$ is a truncated closure sum and approaches $\delta(z_1 - z_2)$ as $N \rightarrow \infty$, as is to be expected since the healing length

in a spatially uniform 1D hard core Bose gas varies inversely with particle number [54]. As a result the width of the null around the diagonal $Q_1 = Q_2$ decreases with increasing N , and vanishes in the limit. For $|z_1 - z_2|$ much larger than the healing length, D reduces to the uncorrelated density product $n(z_1)n(z_2)$, so the spatial extent of the pair distribution function is that of the density and varies as $N^{1/2}$ [67]. Detailed gray-scale plots of $D(z_1, z_2)$ in the (Q_1, Q_2) plane for the cases $N = 2$, $N = 6$, and $N = 10$ are shown in Fig. 1 of [63].

The reduced single-particle density matrix with normalization $\int \rho_1(z, z) dz = N$ is

$$\begin{aligned} \rho_1(z, z') &= N \int \psi_{B0}(z, z_2, \dots, z_N) \\ &\quad \times \psi_{B0}(z', z_2, \dots, z_N) dz_2 \dots dz_N \quad . \end{aligned} \quad (60)$$

For $z \neq z'$ it cannot be expressed in terms of $|\psi_{B0}|^2$, and is therefore very different from that of the ideal Fermi gas. The multi-dimensional integral cannot be evaluated analytically, but in [63] it was evaluated numerically by Monte Carlo integration for not too large values of N , and grayscale plots are shown in Fig. 2 of [63]. More accurate numerical results were found in [68], and highly accurate results for large values of N were found in [69]. In a macroscopic system, the presence or absence of BEC is determined by the behavior of $\rho_1(z, z')$ as $|z - z'| \rightarrow \infty$. Off-diagonal long-range order is present if the largest eigenvalue of ρ_1 is macroscopic (proportional to N), in which case the system exhibits BEC and the corresponding eigenfunction, the condensate orbital, plays the role of an order parameter [70, 71]. Although this criterion is not strictly applicable to mesoscopic systems, if the largest eigenvalue of ρ_1 is much larger than one then it is reasonable to expect that the system will exhibit some BEC-like coherence effects. Thus we examine here the spectrum of eigenvalues λ_j and associated eigenfunctions $\phi_j(z)$ ("natural orbitals") of ρ_1 . The relevant eigensystem equation is

$$\int_{-\infty}^{\infty} \rho_1(z, z') \phi_j(z') dz' = \lambda_j \phi_j(z) \quad . \quad (61)$$

λ_j represents the occupation of the orbital ϕ_j , and one has $\sum_j \lambda_j = N$. Accurate values of the λ_j have been determined in [69]. In particular, the largest eigenvalue λ_1 was shown to be of order \sqrt{N} for large N , as in the spatially uniform case.

Next we examine the momentum distribution, which can be shown [63] to be a double Fourier transform of ρ_1 :

$$N(k) = (2\pi)^{-1} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' \rho_1(z, z') e^{-ik(z-z')} \quad . \quad (62)$$

The spectral representation of the density matrix then leads to $N(k) = \sum_j \lambda_j |\mu_j(k)|^2$ where the μ_j are Fourier transforms of the natural orbitals: $\mu_j(k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \phi_j(z) e^{-ikz} dz$. The key features are that

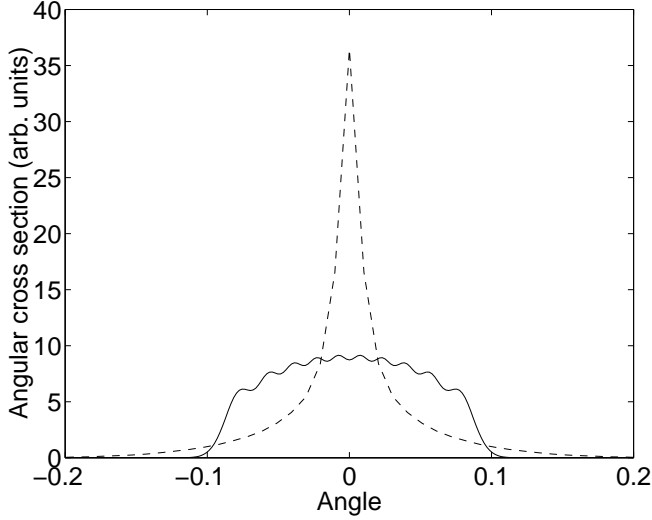


FIG. 2: Angular cross section versus angle $\sin(\theta) \approx \theta$ for $N = 10$. The dashed line is for the 1D gas of impenetrable bosons and the solid line is for the corresponding system of non-interacting fermions

the momentum spectrum maintains the sharp peaked structure reminiscent of the spatially uniform case [16, 59, 60] and that the peak becomes sharper with increasing atom number N . By way of contrast, for a 1D Fermi gas the corresponding momentum spectrum is a filled Fermi sea and can be expressed as $N(k) = \sum_{j=1}^N |\mu_j(k)|^2$. In a recent paper [72] we devised a scheme to measure the momentum spectrum based on Raman outcoupling and showed that the angular cross section accurately mirrors the momentum distribution. Figure 2 shows the angular cross section versus angle for both $N = 10$ impenetrable bosons (dashed line) and the corresponding system of non-interacting fermions (solid line); see [72] for details.

B. Magnetically trapped, spin-aligned fermions

Consider first the case $N = 2$. The spins are frozen in the configuration $\uparrow\uparrow$ by the magnetic field, so the spatial relative wave function $\psi_F(z)$ is antisymmetric (odd in $z = z_1 - z_2$). The Hamiltonian \hat{H}_{1D}^F and corresponding odd-wave pseudopotential were derived in Sec. II B 2. Defining a mapped bosonic (even in z) wave function by $\psi_B(z) = \text{sgn}(z)\psi_F(z)$ and mapped scattering length $a_{1D}^B = a_{1D}^F \equiv a_{1D}$ where $\text{sgn}(z)$ is $+1$ if $z > 0$ and -1 if $z < 0$, one finds that ψ_B satisfies the usual bosonic contact condition $\psi_B'(0+) = -\psi_B'(0-) = -(a_{1D}^B)^{-1}\psi_B(0\pm)$, the zero-range limit $z_0 \rightarrow 0+$ of Eq. (43). Since the kinetic energy contributions from $z \neq 0$ also agree, one has a mapping from the fermionic to bosonic problem which preserves energy eigenvalues and dynamics. The relationship between coupling constants g_{1D}^F in \hat{H}_{1D}^F and g_{1D}^B in $\hat{H}_{1D}^B = -(\hbar^2/2\mu)\partial_z^2 + g_{1D}^B\delta(z)$ is $g_{1D}^B = \hbar^4/\mu^2 g_{1D}^F$, and by (47) this relationship agrees with the low-energy limit of

Eq. (25) of [21, 34]. In the limit $g_{1D}^B = +\infty$ arising when $V_p \rightarrow 0-$, this is the $N = 2$ case of the original mapping [1, 2] from hard sphere bosons to an ideal Fermi gas, but now generalized to arbitrary coupling constants and used in the inverse direction. This generalizes to arbitrary N : Antisymmetric fermionic solutions $\psi_F(z_1, \dots, z_N)$ are mapped to symmetric bosonic solutions $\psi_B(z_1, \dots, z_N)$ via Eqs. (55) and (56). The Fermi contact conditions are $\psi_F|_{z_j=z_\ell+} = -\psi_F|_{z_j=z_\ell-} = -(a_{1D}/2)(\partial_{z_j} - \partial_{z_\ell})\psi_F|_{z_j=z_\ell\pm}$ and imply the Bose contact conditions $(\partial_{z_j} - \partial_{z_\ell})\psi_B|_{z_j=z_\ell+} = -(\partial_{z_j} - \partial_{z_\ell})\psi_B|_{z_j=z_\ell-} = -(2/a_{1D})\psi_B|_{z_j=z_\ell}$, and these are the usual LL contact conditions [23]. The fermionic Hamiltonian is $\hat{H}_{1D}^F = -(\hbar^2/2\mu)\sum_{j=1}^N \partial_{z_j}^2 + g_{1D}^F \sum_{1 \leq j < \ell \leq N} \delta'(z_{j\ell})\hat{\partial}_{j\ell}$ where $\hat{\partial}_{j\ell}\psi = (1/2)[\partial_{z_j}\psi|_{z_j=z_\ell+} - \partial_{z_\ell}\psi|_{z_j=z_\ell-}]$. Although well-defined in the exact Schrödinger equation and in first-order perturbation theory, this fermionic pseudopotential becomes ambiguous in higher-order perturbation theory. However, after mapping to the bosonic Hilbert space one has the usual Lieb-Liniger interaction $g_{1D}^B\delta(z_{j\ell})$ which is well-behaved in all perturbation orders and in second quantization. This generalization of the Fermi-Bose mapping theorem, due to Cheon and Shigehara [26], extends the useful domain of the mapping of Eqs. (55) and (56) to the whole range of coupling constants g_{1D}^B and g_{1D}^F . The first application to the spin-aligned Fermi gas is due to Blume and Granger, who were led to the mapping by consideration of the zero-range limit of a K-matrix formulation [21]. They treated only the case $N = 2$ but did not restrict themselves to the low-energy limit considered here.

The exact ground state [23] of \hat{H}_{1D}^B is known for all positive g_{1D}^B if no external potential or nonzero range interactions are present, and the mapping then generates the exact N -body ground state of \hat{H}_{1D}^F . The dimensionless bosonic and fermionic coupling constants $\gamma_B = mg_{1D}^B/n\hbar^2$ and $\gamma_F = mg_{1D}^F n/\hbar^2$ introduced in Sec. II B 2 satisfy $\gamma_B\gamma_F = 4$. The energy per particle ϵ is related to a dimensionless function $e(\gamma)$ available online [73] via $\epsilon = (\hbar^2/2m)n^2e(\gamma)$ where γ is related to γ_F herein by $\gamma = \gamma_B = 4/\gamma_F$. This is plotted as a function of γ_F in Fig. 3. For $g_{1D,F} \rightarrow \infty$ as occurs at a p-wave Feshbach resonance, one has the “fermionic TG gas” discussed in Sec. II B 2 and [35]. For bosons the TG regime, which maps to the *ideal Fermi* gas, is reached when g_{1D}^B is large enough and/or the density n low enough that $\gamma_B \gg 1$. A similar simplification occurs in the fermionic case, where a fermionic TG regime is reached when g_{1D}^F is large enough and/or n high enough that $\gamma_F \gg 1$. The corresponding fermionic TG gas then maps to the *ideal Bose* gas since $\gamma_B\gamma_F = 4$.

C. Optically trapped, spin-free fermions

In this section we assume that the Hamiltonian is spin-independent and that the fermionic vapor is trapped in a tight *optical* atom waveguide. The spins are then free

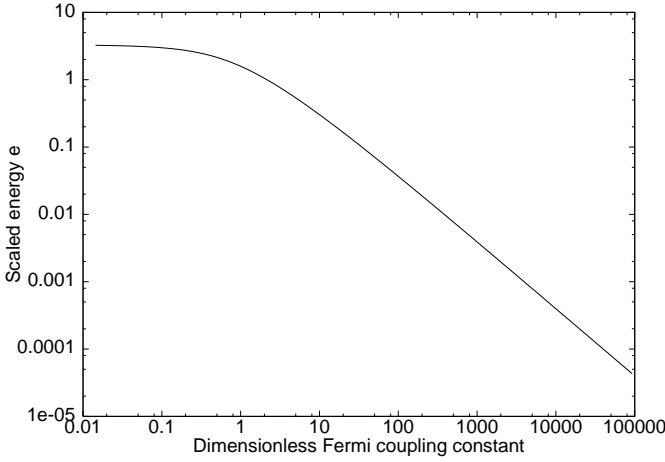


FIG. 3: Log-log plot of scaled ground state energy per particle $e = 2m\epsilon/\hbar^2 n^2$ for the spin-aligned Fermi gas, versus dimensionless fermionic coupling constant γ_F .

to assume whatever configuration minimizes the ground-state energy. First assume $N = 2$. The two-body relative wave functions are $\psi_F(z) = \psi_F^e(z) + \psi_F^o(z)$ where the spatially even part $\psi_e(z)$ contains an implicit spin-odd singlet spin factor, and the spatially odd part $\psi_o(z)$ contains implicit spin-even triplet spin factors. States of combined space-spin bosonic symmetry can be defined by the same mapping $\psi_B(z) = \text{sgn}(z)\psi_F(z)$ used in the previous section, which now maps the spatially even fermionic function ψ_F^e to a spatially odd bosonic function ψ_B^o and the spatially odd fermionic function ψ_F^o to a spatially even bosonic function ψ_B^e while leaving the spin dependence unchanged. Then the even-wave contact conditions for $a_{1D,B}^e$ follow from the odd-wave contact conditions for $a_{1D,F}^o$ and the odd-wave contact conditions for $a_{1D,B}^o$ follow from the even-wave contact conditions for $a_{1D,F}^e$. As before, one has a mapping from the fermionic to bosonic problem which preserves energy eigenvalues and dynamics. The bosonic Hamiltonian is of the same form as the fermionic one (51) but with mapped coupling constants $g_{1D,B}^e = \hbar^4/\mu^2 g_{1D,F}^o$ and $g_{1D,B}^o = \hbar^4/\mu^2 g_{1D,F}^e$. This generalizes to arbitrary N : Fermionic solutions $\psi_F(z_1, \sigma_1; \dots; z_N, \sigma_N)$ are mapped to bosonic solutions $\psi_B(z_1, \sigma_1; \dots; z_N, \sigma_N)$ by the usual mapping (55), (56), where the spin z -component variables σ_j take on the values \uparrow and \downarrow . The N -fermion and N -boson Hamiltonians are both of the form $\hat{H}_{1D} = -(\hbar^2/2m) \sum_{j=1}^N \partial_{z_j}^2 + \sum_{1 \leq j < \ell \leq N} [g_{1D}^e \delta_{j\ell} + g_{1D}^o \delta'(z_{j\ell}) \delta_{j\ell}]$. On fermionic states ψ_F , g_{1D}^e and g_{1D}^o are $g_{1D,F}^e$ and $g_{1D,F}^o$, whereas on the mapped bosonic states $\psi_B = A\psi_F$ they are $g_{1D,B}^e = \hbar^4/\mu^2 g_{1D,F}^o$ and $g_{1D,B}^o = \hbar^4/\mu^2 g_{1D,F}^e$. This is discussed in more detail in [49].

Assume that both $g_{1D,F}^e \geq 0$ and $g_{1D,F}^o \geq 0$. If $g_{1D,F}^o$ is zero then it follows from a theorem of Lieb and Mattis [74] that the fermionic ground state has total spin $S = 0$ (assuming N even), as shown in the spatially uniform case by Yang [75] and with longitudinal trapping by As-trakharchik *et al.* [46]. If $g_{1D,F}^o$ is not negligible then the

ground state may not have $S = 0$. In fact, if $g_{1D,F}^e$ is zero then one can apply a theorem of Eisenberg and Lieb [76] to the mapped spinor boson Hamiltonian, with the conclusion that one of the degenerate Bose ground states is totally spin-polarized, has $S = N/2$, and is the product of a symmetric spatial wave function ψ_{B0} and a symmetric spin wave function. The ground state is then the same as the one discussed in the previous section, except that now there is an $(N+1)$ -fold directional degeneracy since S_z can range from $-N/2$ to $N/2$. Any $S = 0$ state has a higher energy; in fact, for $N > 2$ the mapped Bose gas is partially space-antisymmetric, raising its energy by the exclusion principle.

So far we have considered only the extreme cases where either the even-wave or odd-wave coupling constant vanishes. Assume now that they may take on any non-negative values. Consider first the case $N = 2$ of a longitudinally trapped spinor Fermi gas, with relative spatial wave function $\psi_F(z)$ and Hamiltonian differing from Eq. (51) by addition of a harmonic trap potential $\frac{1}{2}\mu\omega_{\text{long}}^2 z^2$. The even and odd-wave coupling constants are $g_{1D,F}^e = -\hbar^2/\mu a_{1D,F}^e$ and $g_{1D,F}^o = -\hbar^2 a_{1D,F}^o/\mu$ as before. ψ_F may be taken to be either spatially even with associated singlet spin function $\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ which has $S = 0$, or else spatially odd with associated spin function which is one of the $S = 1$ triplets $\uparrow\uparrow, \downarrow\downarrow$, or $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$. The singlet case has spatially even wave functions identical with those of trapped $N = 2$ bosons. The odd-wave pseudopotential then projects to zero so the $S = 0$ eigenstates are independent of $g_{1D,F}^o$, and the even-wave pseudopotential reduces to $g_{1D,F}^e \delta(z)$. The exact eigenstates are known [77], being of the form $D_\nu(|\xi|)$ where D_ν is a Weber (parabolic cylinder) function [78, 79] and $z = \xi\sqrt{\hbar/2\mu\omega_{\text{long}}}$. The absolute value in the argument leads to a cusp at $z = 0$ and the LL cusp condition of Eq. (44) and [23] (with $g_{1D,B}$ replaced by $g_{1D,F}^e$) leads to a transcendental equation for the allowed values of ν : $\Gamma(\frac{1}{2} - \frac{1}{2}\nu)/\Gamma(-\frac{1}{2}\nu) = -\lambda$ in terms of the dimensionless parameter $\lambda = g_{1D,F}^e/2\hbar\sqrt{\mu/\hbar\omega_{\text{long}}}$. The energy eigenvalues are $E(\nu) = (\nu + \frac{1}{2})\hbar\omega_{\text{long}}$, the ground state is that solution for which ν vanishes as $\lambda \rightarrow 0$, and its energy E_0 is a monotonically increasing function of g_{1D}^e . Next consider the $S = 1$ (triplet) solutions, for which ψ_F is spatially odd. Then the even-wave pseudopotential projects to zero, and on carrying out the Fermi-Bose mapping $\psi_B(z) = \text{sgn}(z)\psi_F(z)$ the odd-wave pseudopotential is changed to $g_{1D,B}^e \delta(z)$ with $g_{1D,B}^e = \hbar^4/\mu^2 g_{1D,F}^o$. Thus the $S = 1$ ground state energy is the same function of $g_{1D,B}^e$ that the $S = 0$ ground state energy is of $g_{1D,F}^e$, and is therefore a monotonically decreasing function of $g_{1D,F}^o$. It follows that the $S = 0$ and $S = 1$ ground-state energies are equal on the hyperbola $g_{1D,F}^e g_{1D,F}^o = \hbar^4/\mu^2$ in the $(g_{1D,F}^e, g_{1D,F}^o)$ plane, which forms a phase boundary between the region where the absolute ground state has $S = 0$ and that where it has $S = 1$. For $g_{1D,F}^e g_{1D,F}^o < \hbar^4/\mu^2$ (below the phase boundary) the ground state has $S = 0$ and its energy

is independent of $g_{1D,F}^o$, and for $g_{1D,F}^e g_{1D,F}^o > \hbar^4/\mu^2$ (above the phase boundary) it has $S = 1$ and its energy is independent of $g_{1D,F}^e$.

Since it depends only on a symmetry argument and is independent of ω_{long} , it is reasonable to conjecture that the above phase boundary is valid for all N and for both spatially uniform and longitudinally trapped spinor Fermi gases. In other words we make the following

Conjecture. The ground state of a spinor one-dimensional Fermi gas undergoes a quantum phase transition at $\gamma_e \gamma_o = 4$, being paramagnetic ($S = 0$) for $\gamma_e \gamma_o < 4$ and ferromagnetic, with $S = N/2$ for $\gamma_e \gamma_o > 4$; here γ_e and γ_o are the dimensionless even and odd-wave coupling constants $\gamma_e = mg_{1D,F}^e/n\hbar^2$ and $\gamma_o = mg_{1D,F}^o/n\hbar^2$.

As a first step in motivating this conjecture, it is convenient to change from the representation in terms of space-spin antisymmetric wave functions $\psi_F(z_1, \sigma_1; \dots; z_N, \sigma_N)$ to a representation in which states with $S_z = k - \frac{1}{2}N$ are represented by functions $\varphi_F(x_1, \dots, x_k; y_1, \dots, y_{N-k}) \uparrow_1 \dots \uparrow_k \downarrow_1 \dots \downarrow_{N-k}$, where we assume N even and $1 \leq k \leq N$. φ_F is antisymmetric under permutations of the z -coordinates (x_1, \dots, x_k) of up-spin atoms and also under permutations of those (y_1, \dots, y_{N-k}) of down-spin atoms, but it has no particular symmetry under exchanges $x_i \leftrightarrow y_j$ of up-spin with down-spin atoms. Formally treating up and down-spin particles as different species can be shown [2, 80] to be physically equivalent to the usual representation $\psi_F(z_1, \sigma_1; \dots; z_N, \sigma_N)$. The Hamiltonian is similar to that of Eq. (3) of [46], but also includes odd-wave interactions:

$$\begin{aligned} \hat{H}_F = & -\frac{\hbar^2}{2m} \left(\sum_{i=1}^k \partial_{x_i}^2 + \sum_{j=1}^{N-k} \partial_{y_j}^2 \right) \\ & + \sum_{1 \leq p < q \leq k} \hat{v}_{1D,F}^o(x_p, x_q) + \sum_{1 \leq p < q \leq N-k} \hat{v}_{1D,F}^o(y_p, y_q) \\ & + \sum_{i=1}^k \sum_{j=1}^{N-k} [\hat{v}_{1D,F}^e(x_i, y_j) + \hat{v}_{1D,F}^o(x_i, y_j)] \end{aligned} \quad (63)$$

where the even and odd-wave pseudopotential operators are those of Eq.(51) expressed in terms of relative coordinates $x_p - x_q$, etc. Even-wave interactions $\hat{v}_{1D,F}^e(x_p, x_q)$ and $\hat{v}_{1D,F}^e(y_p, y_q)$ between like-spin particles are absent because φ_F is antisymmetric in both (x_1, \dots, x_k) and in (y_1, \dots, y_{N-k}) , so the corresponding even-wave interactions project to zero. However, both even and odd-wave interactions between up and down-spin particles are present because the states φ_F are neither symmetric nor antisymmetric under exchanges $x_i \leftrightarrow y_j$ between up and down-spin atoms.

The like-spin odd-wave interactions $\hat{v}_{1D,F}^o(x_p, x_q)$ and $\hat{v}_{1D,F}^o(y_p, y_q)$ can be transformed into much simpler even-wave interactions by a generalization of the previous Fermi-Bose mapping. Define a mapped wave function φ_B of a *two-component Bose gas* by $\varphi_B(x_1, \dots, x_k; y_1, \dots, y_{N-k}) =$

$A(x_1, \dots, x_k)A(y_1, \dots, y_{N-k})\varphi_F(x_1, \dots, x_k; y_1, \dots, y_{N-k})$ where A is the previously-defined “unit antisymmetric function” of Eq. (55), equal to ± 1 everywhere, antisymmetric in its arguments, and changing sign here only at “same-species” collisions $x_p = x_q$ and $y_p = y_q$. Then φ_B satisfies a mapped Schrödinger equation $\hat{H}_B \varphi_B = E \varphi_B$ with the same eigenvalue E and with

$$\begin{aligned} \hat{H}_B = & -\frac{\hbar^2}{2m} \left(\sum_{i=1}^k \partial_{x_i}^2 + \sum_{j=1}^{N-k} \partial_{y_j}^2 \right) \\ & + \sum_{1 \leq p < q \leq k} g_{1D,B}^e \delta(x_p - x_q) \\ & + \sum_{1 \leq p < q \leq N-k} g_{1D,B}^e \delta(y_p - y_q) \\ & + \sum_{i=1}^k \sum_{j=1}^{N-k} [\hat{v}_{1D,F}^e(x_i, y_j) + \hat{v}_{1D,F}^o(x_i, y_j)] \end{aligned} \quad (64)$$

Now there are only interactions of LL type [23] between like-spin atoms, and these are straightforward to treat; they have a mapped coupling constant $g_{1D,B}^e = \hbar^4/\mu^2 g_{1D,F}^o$. On the other hand, the mapping does not affect the contact condition for collisions $x_i = y_j$ of unlike-species atoms, so these interactions are the same as in Eq. (63) and retain the original fermionic coupling constants $g_{1D,F}^e$ and $g_{1D,F}^o$.

Consider now the Gross-Pitaevskii (GP) approximation, the variational energy with a trial function $\varphi_B^{GP} = [\prod_{i=1}^k u_\uparrow(x_i)][\prod_{j=1}^{N-k} u_\downarrow(y_j)]$. In the absence of longitudinal trapping one has $u_\uparrow(x_i) = u_\downarrow(y_j) = L^{-1/2}$ where L is the length of the periodic box. When acting on these trivial wave functions the odd-wave interactions $\hat{v}_{1D,F}^o(x_i, y_j)$ in (64) project to zero and the even-wave interactions $\hat{v}_{1D,F}^e(x_i, y_j)$ reduce to simple Lieb-Liniger interactions $g_{1D,F}^e \delta(x_i - y_j)$, so the variational energy per particle in the thermodynamic limit ($N \rightarrow \infty, L \rightarrow \infty, N/L \rightarrow n$) is

$$\epsilon_0^{GP} = n[\alpha^2(g_{1D,B}^e - g_{1D,F}^e) + \frac{1}{4}(g_{1D,B}^e + g_{1D,F}^e)] \quad (65)$$

where $\alpha = S_z/N$ with $S_z = k - \frac{1}{2}N$. Thus ϵ_0^{GP} increases with S_z if $g_{1D,B}^e > g_{1D,F}^e$ and decreases with S_z if $g_{1D,B}^e < g_{1D,F}^e$. It follows that the ground state has $S_z = 0$ if $g_{1D,B}^e > g_{1D,F}^e$ or equivalently $g_{1D,F}^e g_{1D,F}^o < \hbar^4/\mu^2$, and $S = S_z = \frac{N}{2}$ (the maximal value) if $g_{1D,F}^e g_{1D,F}^o > \hbar^4/\mu^2$. This is the same result obtained previously for the exact $N = 2$ ground state in a trap. The ground state has $S_z = 0$ if $\gamma_e \gamma_o < 4$ and $S = S_z = \frac{N}{2}$ if $\gamma_e \gamma_o > 4$, so there is a hyperbolic phase boundary $\gamma_e \gamma_o = 4$ in the (γ_e, γ_o) plane. In the $S = \frac{N}{2}$ ($\alpha = \frac{1}{2}$) phase the ground state energy depends only on $g_{1D,B}^e$, hence on γ_o but not on γ_e , a result which we will show holds also for the *exact* N -atom ground state.

The GP approximation is valid when both $g_{1D,F}^e$ and $g_{1D,B}^e$ are small enough, more precisely when both $\gamma_e \ll 1$

and $\gamma_o \gg 1$. Since the product $\gamma_e \gamma_o$ can be made to assume any desired value without violating these inequalities, it is reasonable to suppose that the phase boundary $\gamma_e \gamma_o = 4$ holds for the *exact* ground state. We now show that this is true. To see this, note first that since φ_B is symmetric in (x_1, \dots, x_k) and in (y_1, \dots, y_{N-k}) , it follows that when acting on φ_B , $g_{1D,B}^e \delta(x_p - x_q)$ is equivalent to $\hat{v}_{1D,B}^e(x_p, x_q)$ defined in connection with Eq. (53) with $g_{1D}^e = g_{1D,B}^e$, and similarly $g_{1D,B}^e \delta(y_p - y_q)$ is equivalent to $\hat{v}_{1D,B}^e(y_p, y_q)$. Furthermore, one may formally add interactions $\hat{v}_{1D,F}^o(x_p, x_q)$ and $\hat{v}_{1D,F}^o(y_p, y_q)$ since these vanish on φ_B because of its symmetry in (x_1, \dots, x_k) and (y_1, \dots, y_{N-k}) . Thus in (64) we may replace $g_{1D,B}^e \delta(x_p - x_q)$ by $[\hat{v}_{1D,B}^e(x_p, x_q) + \hat{v}_{1D,F}^o(x_p, x_q)]$ and $g_{1D,B}^e \delta(y_p - y_q)$ by $[\hat{v}_{1D,B}^e(y_p, y_q) + \hat{v}_{1D,F}^o(y_p, y_q)]$ without changing its action on φ_B . After this has been done, we note that when $g_{1D,F}^e = g_{1D,B}^e$, the resultant Hamiltonian reduces to that of a *one*-component Bose gas with both even and odd-wave interactions, with ground state energy $E_0(S_z)$ which is independent of k , hence independent of S_z . Hence, on the line $\gamma_e \gamma_o = 4$ ground states of all values of S_z from 0 to $N/2$ are degenerate, and it is easy to show that this remains true if spin-independent longitudinal trap potentials are added to \hat{H}_B . To complete the proof we need to convert this into a statement about the total spin S of the ground state, not merely its value of S_z . To do this, first map back to the two-component Fermi gas states $\varphi_F(x_1, \dots, x_k; y_1, \dots, y_{N-k}) \uparrow_1 \dots \uparrow_k \downarrow_1 \dots \downarrow_{N-k}$. They are not in general eigenstates of S , but they can be converted into such eigenstates ψ_F , with the same eigenvalues of both energy and S_z , by antisymmetrizing with respect to all combined space-spin exchanges $(z_i, \sigma_i) \leftrightarrow (z_j, \sigma_j)$ after renaming the variables as follows: $(x_1, \dots, x_k; y_1, \dots, y_{N-k}) \rightarrow (z_1, \dots, z_N)$ and $(\uparrow_1 \dots \uparrow_k \downarrow_1 \dots \downarrow_{N-k}) \rightarrow (\uparrow_1 \dots \uparrow_k \downarrow_{k+1} \dots \downarrow_N)$. In this standard representation the Hamiltonian, total spin, and its z-component are mutually commuting, so nondegenerate energy eigenstates are also eigenstates of \hat{S} and \hat{S}_z , and degenerate ones can be chosen to be such simultaneous eigenstates. Let $\psi_{F0}(\gamma_e, \gamma_o, S_z, S)$ be the lowest such state for given values of γ_e and γ_o . We have shown that if $\gamma_e \gamma_o < 4$ then such a ground state has $S_z = 0$, if $\gamma_e \gamma_o > 4$ it has $S_z = \frac{N}{2}$, and if $\gamma_e \gamma_o = 4$ it can have any value of S_z from 0 to $\frac{N}{2}$. If $\gamma_e \gamma_o > 4$ this state also has $S = \frac{N}{2}$ since $S_z = \frac{N}{2}$ implies $S = \frac{N}{2}$. If $\gamma_e \gamma_o < 4$ then the ground state also has $S = 0$ because if it had $S > 0$ then there would exist states with $0 < |S_z| \leq S$ of the same energy, contradicting the monotonic dependence of energy on S_z demonstrated in (65), which we assume to hold also for the *exact* ground state. It follows that the hyperbola $\gamma_e \gamma_o = 4$ is the boundary between a ground state $S = 0$ phase ($\gamma_e \gamma_o < 4$) and a $S = \frac{N}{2}$ phase ($\gamma_e \gamma_o > 4$).

IV. PROSPECT AND CRITIQUE

It follows from the above that the dependence of the ground-state energy on total spin S is more complicated than envisioned in either [46] or [49]: It is indeed true that if γ_o is *exactly* zero, then the ground state has $S = 0$ as assumed in [46], but if $\gamma_o \neq 0$ then by approaching the confinement-induced resonance (CIR) of γ_e implied by Eq. (45), one can in principle always make $\gamma_e \gamma_o > 4$ even if $0 < \gamma_o \ll 1$, thereby inducing a phase transition from the paramagnetic $S = 0$ phase to the ferromagnetic $S = N/2$ phase. In the opposite case $\gamma_o \gg 1$ encountered near the CIR of γ_o implied by Eq. (47) a phase transition in the opposite direction ($S = N/2$ to $S = 0$) would in principle occur if $\gamma_e \rightarrow 0+$, but in practice there is no way of making γ_e that small. In the region $\gamma_e \gamma_o > 4$ where $S = N/2$ the exact ground state is totally spatially antisymmetric and hence spin-aligned, so its energy is independent of γ_e and given by the results in [49] and Sec. II B 2 herein. In the region $\gamma_e \gamma_o < 4$ where $S = 0$ the ground state is thus far only known for the case $\gamma_o = 0$, where it was determined by Yang [75] in the spatially uniform case and by Astrakharchik *et al.* [46] in the longitudinally trapped case. No analytical or numerical results are yet known for the ground-state energy in the region $\gamma_e \gamma_o < 4$ if $\gamma_o \neq 0$, but it should be investigated by numerical calculations. These will be more complicated than the previous ones [46] due to the presence of both even and odd-wave interactions between up and down-spin atoms. However, if the ground state is real and nodeless [81] they should be feasible, perhaps by using an interaction potential consisting of a narrow and deep well to represent the odd-wave interaction (see Sec. II B 2) together with a somewhat broader “soft rod” potential to represent the even-wave interaction.

Our proof of the exact ground-state phase boundary $\gamma_e \gamma_o = 4$ is a “physicist’s proof” and we make no claim of mathematical rigor. In the first place, as pointed out in Sec. II A 6, a rigorous derivation of the zero-range, 1D limit of the odd-wave interaction between fermions in tight waveguides does not exist at present, although we believe that Eq. (47) is a correct zero-range, low-energy consequence of the K-matrix treatment of Granger and Blume [21]. Furthermore, for $N > 2$ our proof of the phase boundary is not completely rigorous since we had to assume that the ground state energy is a single-valued function of S_z both for $\gamma_e \gamma_o < 4$ and for $\gamma_e \gamma_o > 4$. We feel that this is justified since we proved it to be true in the GP approximation and showed that the criteria for validity of the GP approximation can be satisfied on the phase boundary $\gamma_e \gamma_o = 4$. However, a truly rigorous proof of this phase boundary does not yet exist, and we hope that one will be forthcoming.

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